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AN ALGORITHM FOR ISENTROPIC FLOW

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SUMMARY

An efficient algorithm is presented for the solution of the equations of isentropic gas dynamics with a general convex gas law. The scheme is based on solving linearized Riemann problems approximately, and in more than one dimension incorporates operator splitting. In particular, only two function evaluations in each computational cell are required. The scheme is applied to a standard test problem in gas dynamics for a polytropic gas.

KEY WORDS Isentropic flow

1. INTRODUCTION

In 1988 Glaister[1] proposed an approximate linearized Riemann solver for the Euler equations of gas dynamics for non-ideal gases in one dimension. The scheme has good shock-capturing properties and has proved successful in its application to some standard test problems. There are applications, however (e.g. the flow of natural gas in a pipe), where it is not necessary to use the full Euler equations but to assume that the flow is isentropic. In such situations it is appropriate to use a simplified Riemann solver for the reduced set of equations. We seek here to devise an efficient scheme that has good shock-capturing properties and applies to the equations of isentropic flow for any convex gas law.

In Section 2 we consider the Jacobian matrix of the flux functions for the equations of isentropic flow with a general gas law, and in Section 3 derive an approximate Riemann solver for the solution of these equations. Finally, in Section 4 we display the numerical results achieved for a standard test problem in gas dynamics.

2. EQUATIONS OF FLOW

In this section we state the equations of flow considered and give the eigenvalues and eigenvectors of the Jacobian matrix of one of the corresponding flux functions. We discuss the two-dimensional case for simplicity, but the extension to three dimensions is straightforward.

2.1. Equations of motion

The two-dimensional equations of isentropic flow can be written in conservation form as

$$\mathbf{w}_t + \mathbf{f}_x + \mathbf{g}_y = \mathbf{0}, \tag{1}$$

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where

$$\mathbf{w}=(\rho, \rho u, \rho v)^{\mathrm{T}}, \tag{2}$$

$$\mathbf{f}=(\rho u, p+\rho u^2, \rho uv)^{\mathrm{T}}, \tag{3}$$

$$\mathbf{g}=(\rho v, \rho vu, p+\rho v^2)^{\mathrm{T}}. \tag{4}$$

The quantities $(\rho, u, v, p)=(\rho, u, v, p)(x, y, t)$ represent the density, the velocity in the two co-ordinate directions and the pressure at a general position (x, y) in space at time t. In addition we assume that there is a gas law connecting p and ρ written as

$$p=p(\rho). \tag{5}$$

We assume further that the derivative $\mathrm{d}p/\mathrm{d}\rho$ of equation (5) can be determined.

2.2. Jacobian

The Jacobian matrix $\mathbf{A}=\partial\mathbf{f}/\partial\mathbf{w}$ has eigenvalues

$$\lambda_j=u\pm a, u, \quad j=1, \ldots, 3, \tag{6a–c}$$

with corresponding eigenvectors

$$\mathbf{e}_{1,2}=(1, u\pm a, v)^{\mathrm{T}}, \tag{7a, b}$$

$$\mathbf{e}_3=(0, 0, 1)^{\mathrm{T}}, \tag{7c}$$

where the sound speed a is given by

$$a^2=\mathrm{d}p/\mathrm{d}\rho \tag{8}$$

from equation (5). Similar expressions can be found for the Jacobian $\partial\mathbf{g}/\partial\mathbf{w}$.

3. APPROXIMATE RIEMANN SOLVER

In this section we develop an approximate Riemann solver for the equations of isentropic flow in two dimensions with a general convex gas law which incorporates the technique of operator splitting.

We seek to solve equations (1)–(5) approximately using operator splitting, i.e. we solve successively

$$\mathbf{w}_t+\mathbf{f}_x=\mathbf{0} \tag{9a}$$

and

$$\mathbf{w}_t+\mathbf{g}_y=\mathbf{0} \tag{9b}$$

along x- and y-co-ordinate lines respectively. We consider approximate solutions of equation (9a); then a similar analysis will give approximate solutions of equation (9b).

3.1. Wave speeds for nearby states

Following Godunov,[2] we consider the solution at any time to consist of a series of piecewise constant states. Our aim is then to solve each of these linearized Riemann problems approximately.

Consider two (constant) adjacent states $\mathbf{w}_L$ and $\mathbf{w}_R$ (left and right) close to an average state $\mathbf{w}$, at points L and R on an x-co-ordinate line. We assume that we have approximate eigenvectors

$$\mathbf{r}_{1,2}=(1,\ u\pm a,\ v)^T, \tag{10a, b}$$

$$\mathbf{r}_3=(0,\ 0,\ 1)^T \tag{10c}$$

corresponding to the average state $\mathbf{w}$.

We now seek coefficients α_1, α_2 and α_3 such that

$$\Delta\mathbf{w}=\alpha_1\mathbf{r}_1+\alpha_2\mathbf{r}_2+\alpha_3\mathbf{r}_3 \tag{11a–c}$$

to within $O(\Delta^2)$, where $\Delta(\cdot)=(\cdot)_R-(\cdot)_L$.

From equations (11a) and (11c) we obtain

$$\alpha_3=\Delta(\rho v)-v\Delta\rho, \tag{12}$$

but

$$\Delta(\rho U)=\rho\Delta U+U\Delta\rho, \quad U=u \text{ or } v, \tag{13a, b}$$

to within $O(\Delta^2)$, so that

$$\alpha_3=\rho\Delta v. \tag{14}$$

Also, from equations (11a), (11b) and (13) we find that

$$a(\alpha_1-\alpha_2)=\rho\Delta u, \tag{15}$$

and equation (11a) gives

$$\alpha_1+\alpha_2=\Delta\rho. \tag{16}$$

Thus combining equations (15) and (16) we have

$$\alpha_{1,2}=\tfrac{1}{2}[\Delta\rho\pm(\rho/a)\Delta u], \tag{17}$$

together with α_3 from (14).

We have found α_1, α_2 and α_3 such that

$$\Delta\mathbf{w}=\sum_{j=1}^{3}\alpha_j\mathbf{r}_j \tag{18}$$

to within $O(\Delta^2)$, and a routine calculation verifies that

$$\Delta\mathbf{f}=\sum_{j=1}^{3}\lambda_j\alpha_j\mathbf{r}_j \tag{19}$$

to within $O(\Delta^2)$. We are now in a position to construct the approximate Riemann solver.

3.2. Decomposition for general $\mathbf{w}_L$ *and* $\mathbf{w}_R$

Consider the algebraic problem of finding average eigenvalues $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ and $\tilde{\lambda}_3$ and corresponding average eigenvectors $\tilde{\mathbf{r}}_1$, $\tilde{\mathbf{r}}_2$ and $\tilde{\mathbf{r}}_3$ such that relations (18) and (19) hold exactly for arbitrary states $\mathbf{w}_L$ and $\mathbf{w}_R$ not necessarily close. Specifically, we seek averages $\tilde{\rho}$, $\tilde{u}$, $\tilde{v}$ and $\tilde{a}$ in terms of two adjacent states $\mathbf{w}_L$ and $\mathbf{w}_R$ (on an x-co-ordinate line) such that

$$\Delta\mathbf{w}=\sum_{j=1}^{3}\tilde{\alpha}_j\tilde{\mathbf{r}}_j, \tag{20}$$

$$\Delta\mathbf{f}=\sum_{j=1}^{3}\tilde{\lambda}_j\tilde{\alpha}_j\tilde{\mathbf{r}}_j, \tag{21}$$

where

$$\Delta(\cdot)=(\cdot)_{\mathrm{R}}-(\cdot)_{\mathrm{L}}, \tag{22}$$

$$\mathbf{w}=(\rho,\ \rho u,\ \rho v)^{\mathrm{T}}, \tag{23}$$

$$\mathbf{f}(\mathbf{w})=(\rho u,\ p+\rho u^2,\ \rho uv)^{\mathrm{T}}, \tag{24}$$

$$p=p(\rho), \tag{25}$$

$$\tilde{\lambda}_{1,2,3}=\tilde{u}\pm\tilde{a},\ \tilde{u}, \tag{26a–c}$$

$$\tilde{\mathbf{r}}_{1,2}=(1,\ \tilde{u}\pm\tilde{a},\ \tilde{v})^{\mathrm{T}}, \tag{27a, b}$$

$$\tilde{\mathbf{r}}_3=(0,\ 0,\ 1)^{\mathrm{T}}, \tag{27c}$$

$$\tilde{\alpha}_{1,2}=\tfrac{1}{2}[\Delta\rho\pm(\tilde{\rho}/\tilde{a})\ \Delta u], \tag{28a, b}$$

$$\tilde{\alpha}_3=\tilde{\rho}\Delta v. \tag{28c}$$

We note that the solution to this problem is equivalent to seeking an approximation to the Jacobian $\mathbf{A}$, namely $\tilde{\mathbf{A}}$, with eigenvalues $\tilde{\lambda}_i$ and eigenvectors $\tilde{\mathbf{r}}_i$, such that

$$\Delta\mathbf{f}=\tilde{\mathbf{A}}\Delta\mathbf{w}. \tag{29}$$

The first step in the analysis of the above problem is to write out equations (20) and (21) explicitly, namely

$$\Delta\rho=\tilde{\alpha}_1+\tilde{\alpha}_2, \tag{30a}$$

$$\Delta(\rho u)=\tilde{\alpha}_1(\tilde{u}+\tilde{a})+\tilde{\alpha}_2(\tilde{u}-\tilde{a}), \tag{30b}$$

$$\Delta(\rho v)=\tilde{\alpha}_1\tilde{v}+\tilde{\alpha}_2\tilde{v}+\tilde{\alpha}_3, \tag{30c}$$

$$\Delta(\rho u)=\tilde{\alpha}_1(\tilde{u}+\tilde{a})+\tilde{\alpha}_2(\tilde{u}-\tilde{a}), \tag{30d}$$

$$\Delta(p+\rho u^2)=\Delta p+\Delta(\rho u^2)=\tilde{\alpha}_1(\tilde{u}+\tilde{a})^2+\tilde{\alpha}_2(\tilde{u}-\tilde{a})^2, \tag{30e}$$

$$\Delta(\rho uv)=\tilde{\alpha}_1(\tilde{u}+\tilde{a})\tilde{v}+\tilde{\alpha}_2(\tilde{u}-\tilde{a})\tilde{v}+\tilde{\alpha}_3\tilde{u}. \tag{30f}$$

Solving we find that

$$\tilde{u}=\frac{\sqrt{(\rho_{\mathrm{L}})}u_{\mathrm{L}}+\sqrt{(\rho_{\mathrm{R}})}u_{\mathrm{R}}}{\sqrt{(\rho_{\mathrm{L}})}+\sqrt{(\rho_{\mathrm{R}})}}, \tag{31}$$

$$\tilde{\rho}=\sqrt{(\rho_{\mathrm{L}}\rho_{\mathrm{R}})}, \tag{32}$$

$$\tilde{v}=\frac{\Delta(\rho v)-\rho\Delta v}{\Delta\rho}=\frac{\sqrt{(\rho_{\mathrm{L}})}v_{\mathrm{L}}+\sqrt{(\rho_{\mathrm{R}})}v_{\mathrm{R}}}{\sqrt{(\rho_{\mathrm{L}})}+\sqrt{(\rho_{\mathrm{R}})}}, \tag{33}$$

together with

$$\Delta p=\tilde{a}^2\Delta\rho. \tag{34}$$

Equation (34) is readily satisfied by

$$\tilde{a}^2=\frac{\Delta p}{\Delta\rho}=\frac{p(\rho_{\mathrm{R}})-p(\rho_{\mathrm{L}})}{\rho_{\mathrm{R}}-\rho_{\mathrm{L}}},\quad \Delta\rho\neq 0,\quad \rho_{\mathrm{R}}\neq\rho_{\mathrm{L}}, \tag{35a}$$

$$\tilde{a}^2=\frac{\mathrm{d}p}{\mathrm{d}\rho}(\rho),\quad \Delta\rho=0,\quad \rho_{\mathrm{R}}=\rho_{\mathrm{L}}. \tag{35b}$$

By symmetry, similar results hold for the Jacobian $\partial\mathbf{g}/\partial\mathbf{w}$.

Summarizing, we can now apply the Riemann solver given above to the two-dimensional isentropic equations with a general convex gas law using the technique of operator splitting. We can incorporate the results found here, together with any explicit one-dimensional scalar upwind algorithm, and perform a sequence of one-dimensional calculations along computational grid lines in the x- and y-direction in turn. The algorithm along a line $y=$ constant can be described as follows. Suppose at time level n we have data $\mathbf{w}_L$ and $\mathbf{w}_R$ given at either end of the cell (x_L, x_R) (on a line $y=y_0$), then we update $\mathbf{w}$ to time level $n+1$ in an upwind manner. Thus we

$$\text{add} \quad -\frac{\Delta t}{\Delta x}\tilde{\lambda}_j\tilde{\alpha}_j\tilde{\mathbf{r}}_j \quad \text{to} \quad \mathbf{w}_R \quad \text{if} \quad \tilde{\lambda}_j>0$$

or

$$\text{add} \quad -\frac{\Delta t}{\Delta x}\tilde{\lambda}_j\tilde{\alpha}_j\tilde{\mathbf{r}}_j \quad \text{to} \quad \mathbf{w}_L \quad \text{if} \quad \tilde{\lambda}_j<0,$$

where $\Delta x = x_R - x_L$, Δt is the time interval from level n to $n+1$, and $\tilde{\lambda}_j$, $\tilde{\alpha}_j$ and $\tilde{\mathbf{r}}_j$ are given by

$$\begin{aligned}\tilde{\lambda}_{1,2,3} &= \tilde{u}\pm\tilde{a},\ \tilde{u},\\ \tilde{\mathbf{r}}_{1,2} &= (1,\ \tilde{u}\pm\tilde{a},\ \tilde{v})^T,\\ \tilde{\mathbf{r}}_3 &= (0,\ 0,\ 1)^T,\\ \tilde{\alpha}_{1,2,3} &= \tfrac{1}{2}(\Delta\rho\pm\tilde{\rho}\Delta u/\tilde{a}),\ \tilde{\rho}\Delta v,\end{aligned}$$

with

$$\tilde{\rho}=\sqrt{(\rho_L\rho_R)}, \qquad \tilde{U}=\frac{\sqrt{(\rho_L)}\,U_L+\sqrt{(\rho_R)}\,U_R}{\sqrt{(\rho_L)}+\sqrt{(\rho_R)}}, \quad U=u \text{ or } v,$$

$\tilde{a}^2$ given by equations (35) and $\Delta(\cdot)=(\cdot)_R-(\cdot)_L$. Similar results apply for updating in the y-direction.

The Riemann solver we have constructed in this section is a conservative algorithm (when incorporated with operator splitting) and has the important one-dimensional shock-recognizing property guaranteed by equations (20) and (21). Furthermore, the algorithm is efficient in the sense that only two function evaluations of the gas law are required in each computational cell.

4. NUMERICAL RESULTS

In this section we give the numerical results achieved for a two-dimensional test problem using the scheme of Section 3.

This two-dimensional test problem is concerned with Mach 3 flow in a tunnel containing a step. The tunnel is 3 units long and 1 unit wide. The step is 0·2 units high and is located 0·6 units from the left-hand end of the tunnel. At the left an inflow boundary condition is applied, and at the right, where the exit velocity is supersonic, all gradients are assumed to vanish. The initial conditions for the gas in the tunnel are given by $(\rho_0, u_0, v_0)=(1{\cdot}4, 3, 0)$ and hence p_0 from the gas law $p_0=p(\rho_0)$. Gas is continually fed in at the left-hand boundary with the flow variables taking the initial values given above.

The gas law chosen is one for a polytropic gas and can be written as

$$p=(\rho/\rho_0)^\gamma,$$

where $\gamma=1{\cdot}4$ so that $p_0=1$.

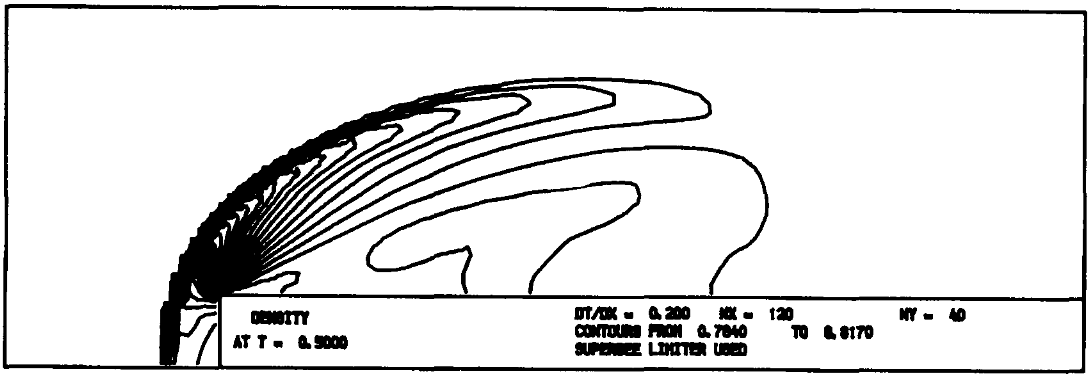

Figure 1. Results for problem at $t = 0{\cdot}5$: bow shock formation

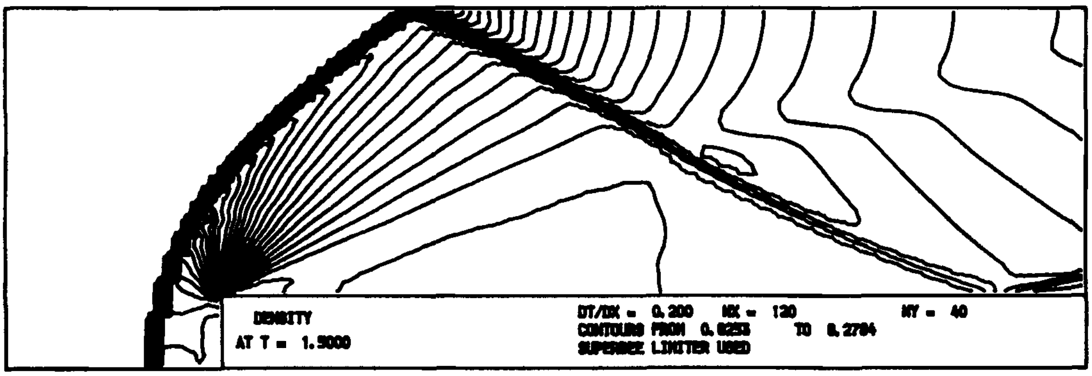

Figure 2. Results for problem at $t = 1{\cdot}5$: reflection at upper wall

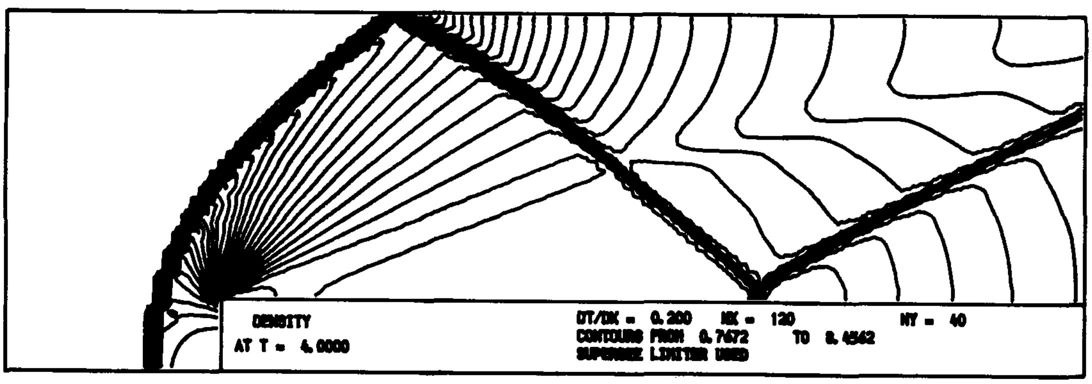

Figure 3. Results for problem at $t = 4{\cdot}0$: reflection at lower wall

Figures 1, 2 and 3 display 31 equally spaced density contours at times $t = 0{\cdot}5$, $1{\cdot}5$ and $4{\cdot}0$ respectively. The figures represent formation of the bow shock, reflection at the upper wall and reflection at the lower wall respectively. A uniform 120×40 mesh was used and the second-order scalar algorithm with the 'superbee' limiter.[3]

(N.B. For both problems we apply a reflection boundary condition at a rigid wall, i.e. we consider an image cell and impose equal density and tangential velocity (for two-dimensional problems), and equal and opposite normal velocity at either end of the cell.)

5. CONCLUSIONS

We have presented a simple Riemann solver for the equations of isentropic flow with a general convex gas law. The scheme has the property that only two function calls are required per cell, and has good shock-capturing properties. This results in an efficient algorithm that has produced satisfactory results for a standard test problem in gas dynamics, and is useful when the flow is known to be isentropic and the full Euler equations do not need to be solved.

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